A Gentle Introduction to Harmonic Functions

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Introduction

One of the most important operators in mathematics, physics and engineering is the Laplacian, which we'll denote Δ. In $\mathbb{R}^2$ the definition is

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

while in $\mathbb{R}^3$ it is

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Tha Laplacian takes a scalar function and produces a new scalar function. Given such a function on $\mathbb{R}^3$,

$$\nabla \cdot (\nabla f) = \nabla \cdot \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \Delta f.$$

Because of this, another common notation for the Laplacian is ∇^2. To minimze notational difficulties, we'll asume all scalar functions are defined on $\mathbb{R}^3$ and use only Δ to denote the Laplacian.

Let's take a look at some situations wherein the Laplacian plays a role. The heat equation states that, given an object of uniform thermal conductivity, the temperature T satisfies

$$\Delta T = k \frac{\partial T}{\partial t},$$

where k is a constant and t is time. In electrostatics, the electric potential V satisfies

$$\Delta V = \rho,$$

where ρ is the charge denisty in space. In the study of fluid flow, an incompressible Newtonian liquid of density ρ, viscosity μ, pressure p and velocity field $\mathbf{v}$ satisfies

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) + \nabla p - \mu \Delta \mathbf{v} = \mathbf{f},$$

where $\mathbf{f}$ is the force field acting on the fluid. All of these equations are 'partial differential equations' (or PDEs for short), and solving them in general is impossible. In fact, the existence and uniqueness of solutions to the fluid equation above is unknown—and worth a million dollars for a complete solution! To better understand the complicated equations above, simplifying physical assumptions can bring all of them to a single equation:

$$\Delta f = 0.$$

The equation $\Delta f = 0$ is so important, it has a name: 'Laplace's equation." A smooth function f which solve Laplace's equation is called *harmonic*.

As simple as it looks, Laplace's equation cannot be solved in general. Rather, we can solve it in various situations: within a box, within a cylinder, outside of a sphere, etc. The solutions in different situations require different techniques and a lot of mathematical ideas. For now, we're going to see what can be said about harmonic functions in general, without solving the equation. Surprisingly, we can say a *lot*.

The Mean-Value Property

In this class, we're going to prove one tricky theorem and deduce everything we can from it. It turns out that harmonic functions are always equal to the average of their nearby values. This theorem makes the idea precise:

Theorem (Mean-value property of harmonic functions). *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a harmonic function. Given a point $\mathbf{p}$ in $\mathbb{R}^3$ and a positive number r, let $S(r)$ be the sphere of radius r centered at $\mathbf{p}$. Then*

$$f(\mathbf{p}) = \frac{1}{4\pi r^2} \iint_{S(r)} f \, d\sigma.$$

The right side of this equation is the average value of f on the sphere; the left side is the value of f at the center of the sphere. Notice that the left side doesn't even depend on r. The proof is tricky but nice; it mainly uses calculus ideas and the divergence theorem.

Proof. Let's define the function

$$A(r) = \frac{1}{4\pi r^2} \iint_{S(r)} f \, d\sigma.$$

We want to show that $A(r)$ is constant and equals $f(\mathbf{p})$, so we'll do this in two steps.

Step 1: $A(r)$ is constant. We'll just show that the r derivative is zero! Doing this, however, is the trickiest part of the proof. We can parametrize the sphere by writing $\mathbf{x} = \mathbf{p} + r\mathbf{n}$, where $\mathbf{n}$ is the unit vector from the center of the sphere pointing at $\mathbf{x}$. Furthermore, $\mathbf{n}$ is the unit normal pointing out of the sphere at $\mathbf{x}$. Rewriting $A(r)$ gives us

$$A(r) = \frac{1}{4\pi r^2} \iint_{S(r)} f \, d\sigma = \frac{1}{4\pi r^2} \iint_{S(1)} f(\mathbf{p} + r\mathbf{n}) r^2 \, d\sigma_1 = \frac{1}{4\pi} \iint_{S(1)} f(\mathbf{p} + r\mathbf{n}) \, d\sigma_1,$$

where $d\sigma_1$ denotes the surface element of the unit ball. Compute $A'(r)$ and recognize the directional derivative inside the integral:

$$A'(r) = \frac{1}{4\pi} \iint_{S(1)} \nabla f(\mathbf{p} + r\mathbf{n}) \cdot \mathbf{n} \, d\sigma_1 = \frac{1}{4\pi r^2} \iint_{S(r)} \nabla f(\mathbf{x}) \cdot \mathbf{n} \, d\sigma = \frac{1}{4\pi r^2} \iint_{S(r)} \nabla f \cdot d\mathbf{A}.$$

Let's call the solid unit ball $B(r)$, so that the boundary of $B(r)$ is $S(r)$. The divergence theorem gives us

$$A'(r) = \frac{1}{4\pi r^2} \iiint_{B(r)} \Delta f \, dV = 0,$$

since $\Delta f = 0$. That's the end of step 1.

Step 2: $\lim_{r \to 0} A(r) = f(\mathbf{p})$. For this, we'll use a hand-waving argument (though it could be made rigorous). Since f is continuous, take r *so* small that f is almost constant. Then $f(\mathbf{x}) \approx f(\mathbf{p})$ everywhere on $S(r)$. As $r \to 0$, this approximation becomes exact. Plugging this in gives us

$$\frac{1}{4\pi r^2} \iint_{S(r)} f \, d\sigma \approx \frac{f(\mathbf{p})}{4\pi r^2} \iint_{S(r)} d\sigma = f(\mathbf{p}).$$

Taking the limit $r \to 0$ finishes the proof. □

The proof was hard work, but now we can reap the benefits. As a first corollary, we see that harmonic functions can't have local extrema. A rigorous proof of this would be too much extra work, so let's take it to be 'intuitively obvious' from the previous theorem.

Corollary. *If f is harmonic and has a local maximum or minimum, then f is constant.*

Think about it for a minute, and we can restate the corollary in another way.

Corollary (Maximum principle for harmonic functions). *If f is harmonic in a bounded solid region, then its absolute maximum and minimum occur on the boundary.*

Yet another rethinking of the corollary gives us a uniqueness theorem!

Corollary (Uniqueness of harmonic functions). *If f and g are harmonic functions and $f = g$ on the surface of a bounded region, then $f = g$ inside the region as well.*

Conclusion: What does theory do for us?

Why does a scientist or mathematician care about these results? Consider this:

- Steady-state temperature, electrostatic potential, and pressure in slow-moving viscous fluid are all harmonic functions. It is intuitively clear that no local maxima can occur, but it's nice that intuition is backed by theory.
- If you guess a solution to $\Delta f = 0$ and it matches the boundary conditions, then it's the right answer. (uniqueness prinicple)
- If you numerically solve $\Delta f = 0$ with, say, finite element methods, the approximation works very well. (mean-value property)
- The real and imaginary parts of a complex differentiable function ($f : \mathbb{C} \to \mathbb{C}$) are harmonic; understanding harmonic functions helps understand differentiable functions on the complex plane (and evaluate some ridiculously-complicated integrals with little effort).
- Eigenvalue problems for Δ and its relatives (the elliptic operators) recently led to a proof of the Poincaré conjecture, a major unsolved problem in algebraic topology. (and it earned the mathematician a Fields medal and million-dollar check!)